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Fuzzy Goal Programming Approach for Structural Optimization

S. S. Rao*
Purdue University, West Lafayette, Indiana 47907
and
K. Sundararaju,† B. G. Prakash,‡ and C. Balakrishna†
Aeronautical Development Agency, Bangalore, India

A fuzzy nonlinear goal programming approach is presented for solving multiobjective optimization problems involving vague and imprecise information. Several computational models, including simple additive, weighted additive, and preemptive priority models, are given for the numerical solution of the problem. The methodologies are illustrated with the help of two structural optimization problems involving multiple goals. The solution of the first example is obtained using a graphical procedure whereas the second example is solved using nonlinear programming techniques. Linear membership functions are used in the numerical work for simplicity. The methodologies presented in this work aid in the preliminary design of structural systems involving imprecise and vague information about the goals and/or constraints.

Introduction

LINEAR goal programming has been extensively used in solving decision-making problems involving linear equations and multiple conflicting goals. The goals can be rank ordered depending on their importance to the decision maker. Goal programming attempts to achieve as many of these goals as possible by minimizing deviational variables from the goal levels depending on their relative weights. Linear goal programming algorithms were developed by Charnes et al.,[1] Ignizio,[2] and Zanakis and Gupta.[3] The extension of goal programming to nonlinear optimization problems has also been considered by several authors.[3,4] A fuzzy programming approach for linear programming problems with several objectives was suggested by Zimmerman.[5] Subsequently, several authors proposed different fuzzy goal programming approaches for solving linear goal programming problems involving imprecise statements and information.[6-10]

Narasimhan[6] suggested a method for solving fuzzy linear goal programming problems under the assumption of linear membership functions. His method involves solving a set of 2^k linear programming problems, each containing $3k$ constraints where k denotes the number of goals in the original problem. Hannan[7] indicated a procedure for formulating a fuzzy goal programming problem as an equivalent single linear programming problem with $2k$ goal-related constraints. The problems associated with the definition of fuzzy priorities were discussed in Ref. 7. A brief review of the history and the state of the art in fuzzy multicriteria programming as of 1982 was presented by Ignizio.[8] The distinction between fuzzy goal programming and fuzzy multicriteria formulations was given in Ref. 9. Models are presented by Hannan[7] for the use of fuzzy goal programming with preemptive priorities, with Archimedian weights, and with the maximization of the membership function corresponding to the minimum goal. A methodology based on the use of a nested hierarchy of priorities for each goal was presented by Rubin and Narasimhan.[10] The importance of multiple objectives in the design of practical engineering systems is well established.[11] Methods for the description and optimum design of fuzzy mechanical and structural systems was presented by the first author.[12,13] In Ref. 12, two formulations, one known as the α-cut approach and the other called the λ-formulation, were presented for solving fuzzy mechanical system problems. The λ-formulation was extended in Ref. 13 for handling multiple objectives in the design of fuzzy structural systems. It can be seen from a review of the literature that 1) all of the references deal with the solution of fuzzy linear goal programming problems, 2) the approach was not extended to solve nonlinear fuzzy goal programming problems, and 3) no engineering design application has been considered using fuzzy goal programming.

This paper aims to propose methodologies for solving fuzzy nonlinear goal programming problems and to illustrate them through application to multiobjective structural design problems.

Nonlinear Goal Programming Formulation

A crisp nonlinear goal programming problem can be stated, in the presence of multiple goals or objectives, as follows[14]:

Minimize

$$F(x) = \left[\sum_{i=1}^{k} (d_j^+ + d_j^-)^p \right]^{\frac{1}{p}}, \qquad p \geq 1$$

subject to

$$f_i(x) + d_i^- - d_i^+ = b_i, \qquad i = 1, 2, \ldots, k$$

$$d_i^- d_i^+ = 0, \qquad i = 1, 2, \ldots, k$$

$$d_i^- \geq 0, \; d_i^+ \geq 0, \qquad i = 1, 2, \ldots, k$$

$$g_j(x) \leq 0, \qquad j = 1, 2, \ldots, m$$

$$\ell_j(x) = 0, \qquad j = 1, 2, \ldots, q \tag{1}$$

where k is the number of objective functions (f_i), b_i is the aspiration level (or goal) of the ith objective function, d_i^- and d_i^+ are over $-$ and under $-$ achievements of the goal b_i, g_j are the inequality constraints on the system, ℓ_j are the equality constraints on the system, and x is a $n -$ component design vector. The value of p in Eq. (1) is based on the utility function chosen by the designer. In many design situations, it will be convenient and useful to solve the problem in Eq. (1) by taking $b_i = f_i^* = f_i(x_i^*)$ where x_i^* is the minimum of $f_i(x)$ subject to

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*Professor, School of Mechanical Engineering.
†Engineer.
‡Assistant Project Director.

the constraints $g_j(x) \le 0$, $j = 1, 2, \ldots, m$, and $\ell_j(x) = 0$, $j = 1, 2, \ldots, q$. In this type of selection of the goals, it is not possible to obtain overachievement of the goals, and hence d_i^- need not be defined. Thus the problem of Eq. (1) can be restated as follows:

Minimize

$$F(x) = \left\{ \sum_{i=1}^{k} [f_i(x) - f_i^*]^p \right\}^{\frac{1}{p}}, \qquad p \ge 1$$

subject to

$$-f_i(x) + f_i^* \le 0, \qquad i = 1, 2, \ldots, k$$
$$g_j(x) \le 0, \qquad j = 1, 2, \ldots, m$$
$$\ell_j(x) = 0, \qquad j = 1, 2, \ldots, q \tag{2}$$

The objective functions f_i can be normalized as follows:

$$f_{in} = \left(\frac{f_i - f_i^{\min}}{f_i^{\max} - f_i^{\min}} \right)$$

where $f_i^{\min}$ and $f_i^{\max}$ denote the minimum and maximum permissible values of f_i. This scheme has been used in the numerical examples presented in this paper.

The concept of Pareto-optimum solution is commonly used in characterizing the solutions of a multiobjective optimization problem. A Pareto-optimum point can be defined as follows: Given a continuous vector objective function $f(x) = [f_1(x), f_2(x), \ldots, f_k(x)]^T$: $R^n \to R^k$ and a constraint set $x \in S \subset R^n$, x^* is a Pareto-optimal solution if $x^* \in S$ and for each $x \in S$ with $f(x) \ne f(x^*)$ there is some j for which $f_j(x) > f_j(x^*)$. Since such a solution is noninferior (or nondominated) to any other design, it must lie on the boundary of achievable designs in terms of the objectives f_i.

It is worth noting that different Pareto-optimal solutions can be obtained by varying the values of p and f_i^* in Eq. (2). However, in some cases, the goal programming may not yield a nondominated or Pareto-optimal solution. For example, if a solution of Eq. (2) contains all deviations $d_j^+ = f_i(x) - f_i^*$ as zero, then that solution may be a dominated solution.[14]

In a practical design, when f_i^*, $i = 1, 2, \ldots, k$, cannot be found or easily determined to be used as goals, a set of "reasonable values" can be chosen for b_i based on the experience of the designer. Once these target values or goals are achieved through the solution of the problem in Eq. (2), the designer can specify a set of smaller or tighter values for b_i for the next problem. This process eventually will lead to the determination of the best set of values for the goals b_j. In the present work, only the values of $p = 1$ and 2 are used in Eq. (2). The value of $p = 1$ corresponds to the situation where a sum of the deviations of the objectives from their respective goals is minimized. The value of $p = 2$ refers to the situation where the root mean squared value of the deviations of the objectives from their respective goals is minimized.

Fuzzy Goal Programming Formulation

When fuzzy information is present, the problem stated in Eq. (2) can be formulated as a fuzzy goal programming problem as follows:

Find

$$x = (x_1, x_2, \ldots, x_n)^T$$

to satisfy

$$f_i(x) \lesssim b_i, \qquad i = 1, 2, \ldots, k$$
$$g_j(x) \le 0, \qquad j = 1, 2, \ldots, m$$
$$\ell_j(x) = 0, \qquad j = 1, 2, \ldots, q \tag{3}$$

where the symbol "$\lesssim$" indicates the fuzzification of the goals and is read as "approximately less than or equal to." The ith fuzzy goal $f_i(x) \lesssim b_i$ in Eq. (3) signifies that the designer will be satisfied even for values larger than b_i up to a stated tolerance limit. If t_i denotes the tolerance zone for the goal b_i, the fuzzy goal can be restated in the form of a linear membership function as

$$\mu_{f_i}(x) = \begin{cases} 1 & \text{if} \quad f_i(x) \le b_i \\ \left[\dfrac{-f_i(x) + b_i + t_i}{d_i} \right] & \text{if} \quad b_i \le f_i(x) \le b_i + t_i \\ 0 & \text{if} \quad f_i(x) \ge b_i + t_i \end{cases} \tag{4}$$

Note that the constraints of the original problem, namely, $g_j(x) \le 0$ and $\ell_j(x) = 0$, are assumed to be crisp in Eq. (3). Reference 16 gives the details of fuzzy mathematics. The problem stated in Eq. (3) can be solved using different models as outlined later. The simple additive model is a direct implementation of the model used for fuzzy linear goal programming.[6,7] The weighted and squared additive models are extensions of the simple additive model whereas the preemptive priority model is a new approach used in this work. In addition, the computational models represented by Eqs. (26–29) are new and have not been used earlier.

Simple Additive Model

The simple additive formulation of fuzzy goal programming involves the maximization of the sum of the membership functions of the fuzzy goals:

Maximize

$$F(x) = \sum_{i=1}^{k} \mu_{f_i}(x)$$

subject to

$$\mu_{f_i}(x) = \frac{-f_i(x) + b_i + t_i}{d_i}, \qquad i = 1, 2, \ldots, k$$
$$0 \le \mu_{f_i}(x) \le 1, \qquad i = 1, 2, \ldots, k$$
$$g_j(x) \le 0, \qquad j = 1, 2, \ldots, m$$
$$\ell_j(x) = 0, \qquad j = 1, 2, \ldots, q \tag{5}$$

where $F(x)$ can be considered as the overall fuzzy achievement (or decision or objective) function. The maximization of μ_{f_i} amounts to maximizing the level of satisfaction corresponding to the value of the objective function or goal.

Weighted Additive Model

If achievement of certain goals is more important compared with the others, the fuzzy achievement function in Eq. (5) can be modified to reflect this situation as

$$F(x) = \sum_{i=1}^{k} \omega_i \mu_{f_i}(x) \tag{6}$$

where ω_i denotes the relative importance of satisfying the ith fuzzy goal compared with others.

Squared Additive Model

Analogous to using $p = 2$ in Eq. (2), the fuzzy goal programming problem of Eq. (3) can be formulated using the sum of squares of the membership functions as the fuzzy achievement function:

Maximize

$$F(x) = \sum_{i=1}^{k} \mu_{f_i}^2(x)$$

subject to

$$\mu_{f_i}(x) = \frac{-f_i(x) + b_i + t_i}{d_i}, \qquad i = 1, 2, \ldots, k$$

$$0 \leq \mu_{f_i}(x) \leq 1, \qquad i = 1, 2, \ldots, k$$

$$g_j(x) \leq 0, \qquad j = 1, 2, \ldots, m$$

$$\ell_j(x) = 0, \qquad j = 1, 2, \ldots, q \tag{7}$$

Preemptive Priority Model

In most design problems, the objective functions and hence the goals are noncommensurable, i.e., have different units. In some multiobjective problems, there may be a particular goal or subset of goals that are most important, and other goals cannot be considered unless these goals are achieved. For example, in mechanism design, unless "loop closure" is achieved, other objectives such as structural error and shaking force should not be considered. Similarly, in the design of aircraft for certain types of missions, unless the goal of flutter speed is achieved, the minimization of weight or maximization of frequency should not be considered. In such applications, we need to adopt a preemptive priority structure by assigning a priority factor P_i for the ith goal such that $P_i \gg P_{i+1}$ is satisfied. This implies that $P_i \neq NP_{i+1}$ for any large value of N and the goals of ith priority level have higher priority than the goals of i + 1st priority level. The function to be minimized is formulated as

$$F(x) = \sum_{i=1}^{r} P_i \sum_{j=1}^{p_i} (d_j^- + d_j^+)$$

This makes it possible to achieve the ith level goals to the fullest possible extent before achieving the i + 1st level goals. If the k objective functions or goals are subdivided into r priority levels with $p_1, p_2, \ldots, p_r$ denoting the number of objectives in the priority levels $1, 2, \ldots, r$, respectively, and $p_1 + p_2 + \cdots + p_r = k$, the goal programming problem of Eq. (3) can be solved as a sequence of r subproblems using the simple additive model. For example, in the first subproblem, the p_1 goals belonging to the first priority level only are considered and the problem is solved as follows:

Maximize

$$F_1(x) = \sum_{j=1}^{p_1} \mu_{f_j}$$

subject to

$$g_j(x) \leq 0, \qquad j = 1, 2, \ldots, m$$

and

$$\ell_j(x) = 0, \qquad j = 1, 2, \ldots, q \tag{8}$$

If $\mu_{f_j}^*, j = 1, 2, \ldots, p_1$ denote the membership function values achieved for the first p_1 goals by solving the problem of Eq. (8), the next subproblem is solved by constructing the second fuzzy achievement function as

$$F_2(x) = \sum_{j=p_1+1}^{p_1+p_2} \mu_{f_j}$$

with the constraints $\mu_{f_j}(x) = \mu_{f_j}^*, j = 1, 2, \ldots, p_1$ added along with $g_j(x) \leq 0, j = 1, 2, \ldots, m$, and $\ell_j(x) = 0, j = 1, 2, \ldots, q$. This procedure is continued until the rth subproblem is solved. In general, the problem solved in sth priority level can be stated as follows:

Maximize

$$F_s = \sum_{j=p_1+p_2+\cdots+p_{s-1}+1}^{p_1+p_2+\cdots+p_s} \mu_{f_j}$$

subject to

$$\mu_{f_j}(x) = \mu_{f_j}^*, \quad j = 1, 2, \ldots, (p_1 + p_2 + \cdots + p_{s-1})$$

$$g_j(x) \leq 0, \qquad j = 1, 2, \ldots, m$$

$$\ell_j(x) = 0, \qquad j = 1, 2, \ldots, q \tag{9}$$

where $\mu_{f_j}^*$ denote the membership function values achieved in the solutions of first $(p - 1)$ subproblems.

Illustrative Examples

Example 1

The two-bar truss shown in Fig. 1 is considered as the first example to illustrate the fuzzy goal programming procedure. The design variables are taken as $x_1 = A/A_{\min}$ and $x_2 = x/h$ where A is the cross-sectional area of the members, $A_{\min}$ is the specified minimum permissible value of A, x is the position of the joints 1 and 2, and h is the specified height of the truss. The truss is assumed to be symmetric about the y axis. The coordinates of joint 3 are held constant. The weight of the truss and the total deflection of joint 3 under the given load are treated as objective functions f_1 and f_2. The goals b_1 and b_2 are to be selected based on the designer's experience. One possibility is to use the optimal values of f_1 and f_2 obtained by solving the single objective optimization problems with all crisp constraints. Using this approach, the goals are selected as $b_1 = 36.1493$ lb and $b_2 = 0.0182$ in. The stresses induced in the two members are constrained to be smaller than the permissible stress σ_0. In addition, lower and upper bounds are placed on the design variables. Thus the crisp goal programming problem becomes the following:

Find

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

which minimizes

$$F(x) = \left[\sum_{i=1}^{2} (d_i^+ + d_i^-)^p \right]^{\frac{1}{p}}, \qquad p \geq 1$$

subject to

$$f_i(x) + d_i^- - d_i^+ = b_i, \qquad i = 1, 2$$

$$d_i^+ d_i^- = 0, \qquad i = 1, 2$$

$$d_i^+ \geq 0, d_i^- \geq 0, \qquad i = 1, 2$$

$$g_1(x) = \frac{\sigma_1(x)}{\sigma_0} - 1 \leq 0$$

$$g_2(x) = \frac{\sigma_2(x)}{\sigma_0} - 1 \leq 0$$

$$g_{2+i}(x) = \left[\frac{x_i^{(\ell)}}{x_i} \right] - 1 \leq 0, \qquad i = 1, 2$$

$$g_{4+i}(x) = 1 - \left[\frac{x_i}{x_i^{(u)}} \right] \leq 0, \qquad i = 1, 2 \tag{10}$$

where

$$f_1(x) = 2\rho h x_2 \sqrt{1 + x_1^2} A_{\min} \tag{11}$$

$$f_2(x) = \frac{Ph(1 + x_1^2)^{1.5}(1 + x_1^4)^{0.5}}{2\sqrt{2} E x_1^2 x_2 A_{\min}} \tag{12}$$

$$\sigma_1(x) = \frac{P(1 + x_1)(1 + x_1^2)^{0.5}}{2\sqrt{2} x_1 x_2 A_{\min}} \tag{13}$$

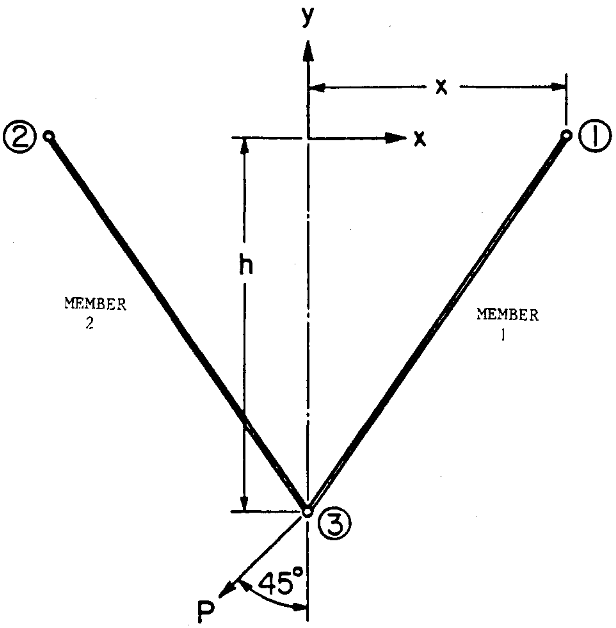

Fig. 1 Two-bar truss.

$$\sigma_2(x) = \frac{P(x_1 - 1)(1 + x_1^2)^{0.5}}{2\sqrt{2}x_1 x_2 A_{\min}} \tag{14}$$

and where E is Young's modulus, ρ is the weight density of the material, P is the applied load, $x_i^{(l)}$ is the lower bound on x_i (i = 1, 2), and $x_i^{(u)}$ is the upper bound on x_i (i = 1, 2). The data are taken as ρ = 0.283 lb/in.^{3}, h = 100 in., $A_{\min}$ = 1 in.^{2}, P = 10,000 lb, $E = 30 \times 10^6$ lb/in.^{2}, σ_0 = 20,000 lb/in.^{2}, $x_1^{(l)} = 0.1$, and $x_2^{(l)} = 1.0$.

For the fuzzy goal programming formulation, the ranges of f_1 and f_2 are taken as 36.1493–186.7361 lb and 0.0182–0.0943 in., respectively. As stated earlier, the limiting values represent the best and worst possible values of the corresponding objective function when the objective functions are minimized individually over the feasible design space. Thus the membership functions of the objectives become

$$\mu_{f_1}(x) = \begin{cases} 1 & \text{if } f_1(x) \le 36.1493 \\ \left[\dfrac{186.7361 - f_1(x)}{186.7361 - 36.1493}\right] & \text{if } 36.1493 < f_1(x) < 186.7361 \\ 0 & \text{if } f_1(x) \ge 186.7361 \end{cases} \tag{15}$$

$$\mu_{f_2}(x) = \begin{cases} 1 & \text{if } f_2(x) \le 0.0182 \\ \left[\dfrac{0.0943 - f_2(x)}{0.0943 - 0.0182}\right] & \text{if } 0.0182 < f_2(x) < 0.0943 \\ 0 & \text{if } f_2(x) \ge 0.0943 \end{cases} \tag{16}$$

The design problem is set up using the fuzzy goal programming formulations outlined earlier. The problems are solved graphically to obtain insight into the qualitative behavior of the optimum solution.

1) Simple additive model: Maximize

$$F = \mu_{f_1} + \mu_{f_2} \tag{17}$$

subject to

$$0.1 \le \mu_{f_i} \le 1, \qquad i = 1, 2$$

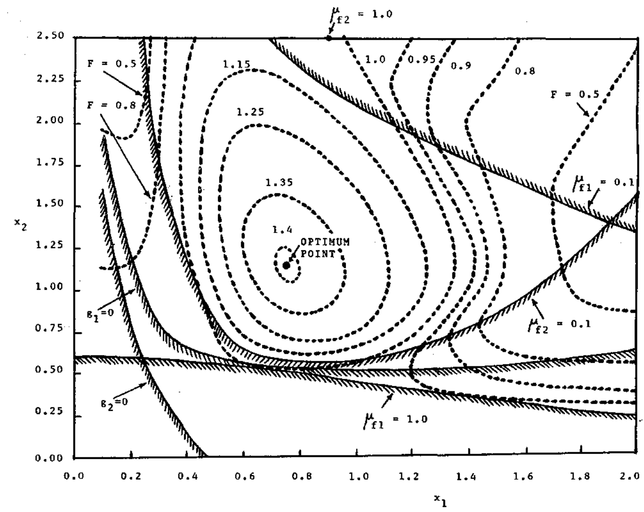

Fig. 2 Graphical optimization using simple additive model.

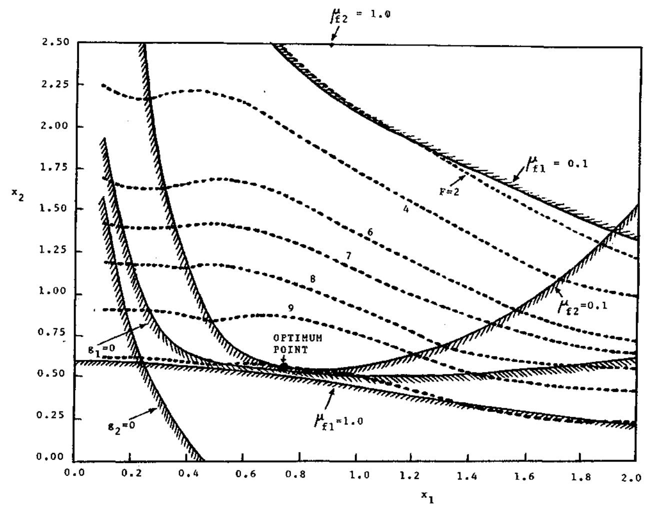

Fig. 3 Graphical optimization using weighted sum model (case *i*).

and

$$g_j(x) \le 0, \qquad j = 1, 2, \ldots, 6 \tag{18}$$

where μ_{f_i} are given by Eqs. (15) and (16), and g_i by Eqs. (10). The constraint boundaries, the contours of the objective function, and the optimum solution of the problem are shown in the design space in Fig. 2.

2) Weighted sum model:

(i) Maximize

$$F = 10\mu_{f_1} + \mu_{f_2} \tag{19}$$

subject to the inequalities of Eq. (18).

(ii) Maximize

$$F = \mu_{f_1} + 10\mu_{f_2} \tag{20}$$

subject to the inequalities of Eq. (18).

The graphical solution of the problem stated in (i) is shown in Fig. 3.

3) Preemptive priority model: The problem is solved in two stages as follows.

Stage (i): Maximize

$$F = \mu_{f_1}$$

subject to

$$0.1 \le \mu_{f_1} \le 1$$

and

$$g_j \le 0, \quad j = 1, 2, \ldots, 6 \tag{21}$$

Stage (ii): Maximize

$$F = \mu_{f_2}$$

subject to

$$0.1 \le \mu_{f_2} \le 1$$

$$0.95\ \mu_{f_1}^* \le \mu_{f_1} \le \mu_{f_1}^*$$

and

$$g_j \le 0, \quad j = 1, 2, \ldots, 6 \tag{22}$$

where $\mu_{f_1}^*$ is the maximum of μ_{f_1} found by solving the problem of Eq. (21).

The results given by the various models are summarized in Table 1.

As an extension of the procedure, the constraints are also assumed to be fuzzy by allowing the maximum permissible stress to include the range σ_0 to $(\sigma_0 + t)$ and the lower and upper bounds on the design variables to have the ranges $[x_i^{(\ell)} - \Delta x_i^{(\ell)}]$ to $x_i^{(\ell)}$ and x_i^u to $[x_i^{(u)} + \Delta x_i^u]$, respectively. The range σ_0 to $\sigma_0 + t$ might represent the band of values observed experimentally for the yield stress over which the design is acceptable. Similarly the ranges on design variables might represent the specified tolerances indicating the limitations of the machining or production process. Thus the membership functions on constraints can be stated as follows:

$$\mu_{g_i}(x) = \begin{cases} 1 & \text{if } \sigma_i(x) \le \sigma_0 \\ \left[\dfrac{\sigma_0 + t - \sigma_i(x)}{t}\right] & \text{if } \sigma_0 < \sigma_i(x) < \sigma_0 + t,\ i = 1,2 \\ 0 & \text{if } \sigma_i(x) \ge \sigma_0 + t \end{cases} \tag{23}$$

$$\mu_{g_{2+i}}(x) = \begin{cases} 1 & \text{if } x_i \ge x_i^{(\ell)} \\ \left[\dfrac{-x_i + x_i^{(\ell)} + \Delta x_i^{(\ell)}}{\Delta x_i^{(\ell)}}\right] & \text{if } x_i^{(\ell)} - \Delta x_i^{(\ell)} < x_i < x_i^{(\ell)},\ i = 1,2 \\ 0 & \text{if } x_i < x_i^{(\ell)} - \Delta x_i^{(\ell)} \end{cases} \tag{24}$$

$$\mu_{g_{4+i}}(x) = \begin{cases} 1 & \text{if } x_i \le x_i^{(u)} \\ \left[\dfrac{x_i^{(u)} + \Delta x_i^{(u)} - x_i}{\Delta x_i^{(u)}}\right] & \text{if } x_i^{(u)} < x_i < x_i^{(u)} + \Delta x_i^{(u)},\ i = 1,2 \\ 0 & \text{if } x_i \ge x_i^{(u)} + \Delta x_i^{(u)} \end{cases} \tag{25}$$

With both the objective functions and all of the six constraints defined as fuzzy quantities, the following problems were solved:

1) Simple additive model: Find x that maximizes

$$\sum_{i=1}^{2} \mu_{f_i} + \sum_{j=1}^{6} \mu_{g_j} \tag{26}$$

subject to

$$0.1 \le \mu_{f_i} \le 1, \quad i = 1, 2$$

and

$$0.1 \le \mu_{g_j} \le 1, \quad j = 1, 2, \ldots, 6 \tag{27}$$

2) Sum of squares model: Find x that maximizes

$$\sum_{i=1}^{2} \mu_{f_i}^2 + \sum_{j=1}^{6} \mu_{g_j}^2 \tag{28}$$

subject to the inequalities of Eq. (27).

Table 1 Results for two-bar truss

Model	Design vector[a]	Objective function	Active behavior constraints	Membership functions, objectives: μ_{f_1}, weight	Membership functions, objectives: μ_{f_2}, deflection
1) Simple additive model	{0.77, 1.12}	1.40158	—	0.70638 (80.4)	0.69520 (0.0413)
2) Weighted sum model					
(i) $F = 10\mu_{f_1} + \mu_{f_2}$	{0.73, 0.55}	9.944	Stress in member 1	0.98411 (38.542)	0.10286 (0.08647)
(ii) $F = \mu_{f_1} + 10\mu_{f_2}$	{0.76, 2.40}	9.9423	—	0.10703 (170.62)	0.98352 (0.01945)
3) Preemptive priotity model					
stage (i)	{0.6743, 0.5295}	1.0	Stress in member 1, upper bound on μ_{f_1}	1.0 (36.1494)	0.0 (0.0943)
stage (ii)	{0.80, 0.55}	0.142	Stress in member 1	0.975 (39.9)	0.142 (0.0835)
4) Formulation including fuzzy constraints					
Eqs. (26) and (27)	{0.77, 1.14}	7.4029	—	0.699 (81.4759)	0.704 (0.0407)
Eqs. (28) and (27)	{0.76, 1.15}	6.9840	—	0.697 (81.7545)	0.7057 (0.0406)
Eqs. (29) and (27)	{0.77, 1.14}	0.49205	—	0.699 (81.4759)	0.704 (0.0407)

[a] $x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

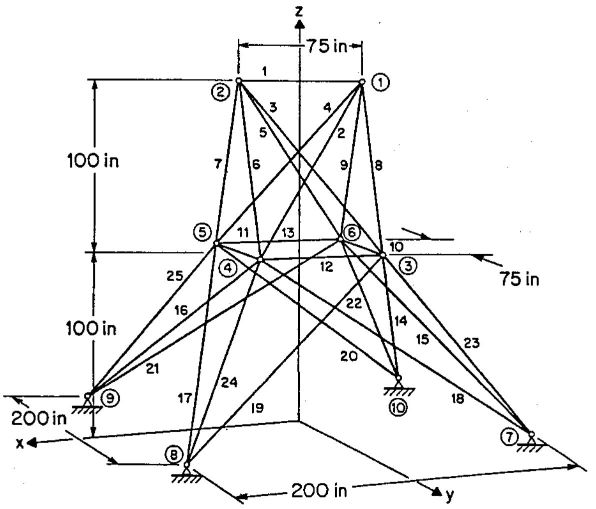

Fig. 4 Twenty-five-bar truss.

Table 2 Loads acting on the 25-bar truss

Load condition no.	Load component	Value at joint, lb 1	2	3	6
1	F_x	0	0	0	0
	F_y	20,000	− 20,000	0	0
	F_z	− 5000	− 5000	0	0
2	F_x	1000	0	500	500
	F_y	10,000	10,000	0	0
	F_z	− 5000	− 5000	0	0

3) Product model: Find $\boldsymbol{x}$ that maximizes

$$\prod_{i=1}^{2} \mu_{f_i} \prod_{j=1}^{6} \mu_{g_j} \tag{29}$$

subject to the inequalities of Eq. (27).

The results given by all of the models 1), 2), and 3) are given in Table 1. It can be observed that the simple additive model (Fig. 2) gave the optimum design vector as ($x_1^* = 0.77$, $x_2^* = 1.12$), which corresponds to a weight of 80.4 lb and a deflection of 0.0413 in. These values can be compared with the best possible values: weight = 36.1493 lb, deflection = 0.0182 in. The weighted sum models yielded better values to the goals having larger weights. For example, when $F = 10\mu_{f_1} + \mu_{f_2}$ is maximized, the optimum point is found to be $\boldsymbol{x} = (0.73, 0.55)^T$ at which the constraints $\mu_{f_2} = 0.1$ and $g_1 = 0$ are active. This corresponds to a weight of 38.542 lb (closer to the best weight of 36.1493 lb) and a deflection of 0.08647 in. Similarly, when $F = \mu_{f_1} + 10\mu_{f_2}$ is maximized, the optimum point gave the values of weight as 170.62 lb and deflection as 0.01945 in. (closer to the best value of 0.0182 in.). In the preemptive priority model, the results were biased by the first priority goal considered. When $F = \mu_{f_1}$ is maximized, the optimum point lies on the constraint boundaries $g_1 = 0$ and $\mu_{f_1} = 1.0$. Subsequently, when $F = \mu_{f_2}$ is maximized, the optimum point moved to the point at which the constraints $\mu_{f_2} = 0.1$, $\mu_{f_1} = 0.95$, and $g_1 = 0$ are active. The weight corresponding to this design is 39.9 lb, which is close to its best value of 36.1493 lb. It can be seen that the simple additive model gave a better compromise result. When constraints were also considered to be fuzzy, all of the three models, namely, the product, the simple additive, and the sum of squares models, yielded essentially the same optimum results. This indicates that any of the three models can be used for constructing the objective function. Another observation is that the results given by these models are almost the same as those given by the simple additive model of Eq. (18).

Example 2

The 25-bar space truss shown in Fig. 4 is considered with three objective functions or goals. This truss is designed such that the stresses induced in the members lie below the yield stress as well as the buckling stresses under the two load conditions specified in Table 2. The cross sections of the members are assumed to be tubular with a diameter to thickness ratio of 100 so that the buckling stress in ith member (σ_{ib}) can be expressed as

$$\sigma_{ib} = -\frac{100.01\pi E A_i}{8\ell_i^2}, \qquad i = 1, 2, \ldots, 25 \tag{30}$$

where E is Young's modulus, A_i are the cross-sectional areas, and ℓ_i are the lengths of the members. Eight independent areas of cross section of the members are treated as design variables:

$$\begin{aligned} x_1 &= A_1 \\ x_2 &= A_j\,(j = 2, 3, 4, 5) \\ x_3 &= A_j\,(j = 6, 7, 8, 9) \\ x_4 &= A_{10} = A_{11} \\ x_5 &= A_{12} = A_{13} \\ x_6 &= A_j\,(j = 14, 15, 16, 17) \\ x_7 &= A_j\,(j = 18, 19, 20, 21) \\ x_8 &= A_j\,(j = 22, 23, 24, 25) \end{aligned} \tag{31}$$

The weight of the structure, the deflection of node 1 in both the load conditions, and the negative of the fundamental natural frequency of vibration are treated as objective functions. The ranges of the fuzzy goals for these objectives are set as follows:

Weight: 233.07–1619.3258 lb
Deflection: 0.3083–1.925 in.
Negative of natural frequency: − 108.6224– − 70.2 Hz

These ranges were based on the best and worst achievable values of the objectives when each objective is optimized individually (without a consideration of the other objectives) over the feasible design space. The membership functions of the objectives are constructed as follows:

$$\mu_{f_1}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_1(\boldsymbol{x}) \le 233.07 \\ \left[\dfrac{-f_1(\boldsymbol{x}) + 1619.3258}{1619.3258 - 233.07}\right] & \text{if } 233.07 < f_1(\boldsymbol{x}) < 1619.3258 \\ 0 & \text{if } f_1(\boldsymbol{x}) \ge 1619.3258 \end{cases} \tag{32}$$

$$\mu_{f_2}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_2(\boldsymbol{x}) \le 0.3083 \\ \left[\dfrac{-f_2(\boldsymbol{x}) + 1.925}{1.925 - 0.3083}\right] & \text{if } 0.3083 < f_2(\boldsymbol{x}) < 1.925 \\ 0 & \text{if } f_2(\boldsymbol{x}) \ge 1.925 \end{cases} \tag{33}$$

$$\mu_{f_3}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_3(\boldsymbol{x}) \le -108.624 \\ \left[\dfrac{-f_3(\boldsymbol{x}) - 70.2082}{+108.6224 - 70.2082}\right] & \text{if } -108.6224 < f_3(\boldsymbol{x}) < -70.2082 \\ 0 & \text{if } f_3(\boldsymbol{x}) \ge -70.2082 \end{cases} \tag{34}$$

The design problem is formulated as a fuzzy goal programming problem using simple additive, weighted sum, sum of squares, and preemptive priority models, and the results are

Table 3 Results for 25-bar truss using additive models

Quantity	Simple additive model $F=\sum_{i=1}^{3}\mu_{f_i}$	Weighted sum model $F=10\mu_{f_1}+\mu_{f_2}+\mu_{f_3}$	Weighted sum model $F=\mu_{f_1}+10\mu_{f_2}+\mu_{f_3}$	Weighted sum model $F=\mu_{f_1}+\mu_{f_2}+10\mu_{f_3}$	Sum of squares $F=\sum_{i=1}^{3}\mu_{f_i}^2$
Design vector $x=\left\{\begin{matrix}x_1\\x_2\\\vdots\\x_8\end{matrix}\right\}$					
	0.10006[a]	0.10000[a]	0.10319[a]	0.10134[a]	0.10017[a]
	0.99866	0.82266	3.7938	0.80056	0.81604
	1.1191	0.73728	4.9486[b]	0.74535	0.7844
	0.67904	0.10637[a]	0.10000[a]	0.71671	0.71116
	0.75505	0.10067[a]	0.10124[a]	0.85990	0.75202
	1.3798	0.55742	2.5682	1.9323	0.92385
	1.7198	0.96033	4.6393	1.7523	1.1427
	4.0146	1.5582	4.9506[b]	3.8134	2.6472
Objective function	2.14864	10.4271	10.4055	10.8783	1.4969
Active behavior constraints	—	Buckling of members 2, 5, 7, 8, 19, and 20 in load case 1 and of member 16 in load case 2	—	Buckling of members 2, 5, 7, and 8 in load case 1	Buckling of members 2 and 5 in load case 1
Membership functions, objectives					
μ_{f_1}, weight	0.76332 (561.106)	0.97226 (271.454)	0.30264 (1199.76)	0.75790 (568.593)	0.88796 (388.326)
μ_{f_2}, deflection	0.52864 (1.07035)	0.14368 (1.69271)	0.97836 (0.343281)	0.34036 (1.37473)	0.25935 (1.50570)
μ_{f_3}, frequency	0.85668 (101.655)	0.56077 (87.2670)	0.31931 (75.5260)	0.97791 (107.554)	0.80072 (98.9341)

[a]Lower bound on design variable.
[b]Upper bound on design variable.

Table 4 Results for 25-bar truss using preemptive priority models

Quantity	Stage (i)[a]	Stage (ii)[b]	Stage (iii)[c]
Design vector $x=\left\{\begin{matrix}x_1\\x_2\\\vdots\\x_8\end{matrix}\right\}$			
	0.18220	0.10113[d]	0.10118[d]
	0.77964	1.0640	0.98769
	0.99726	1.5989	1.3231
	0.10025[d]	0.10051[d]	0.10039[d]
	0.10401[d]	0.10113[d]	0.10031[d]
	0.56889	0.56227	0.55580
	0.98780	0.98246	0.97676
	0.80037	1.1905	1.5164
Objective function	0.99260	0.38489	0.35735
Active behavior constraints	Buckling of members 19 and 20 in load condition 1 and of member 16 in load condition 2	Buckling of members 19 and 20 in load condition 1 and of member 16 in load condition 2	Buckling of members 19 and 20 in load condition 1 and of member 16 in load condition 2
Membership functions, objectives			
μ_{f_1}, weight	0.99260 (243.253)	0.94942 (303.111)[d]	0.94890 (303.845)[d]
μ_{f_2}, deflection	0.103439 (1.75777)[d]	0.38489 (1.30274)	0.36877 (1.32880)[d]
μ_{f_3}, frequency	0.19796 (69.6257)	0.18431 (68.9622)	0.35735 (77.3757)

[a]Max. $F=\mu_{f1}$ subject to $0.1\le\mu_{fi}\le 1$, $i=2, 3$.
[b]Max. $F=\mu_{f2}$ subject to $0.95\mu_{f1}^*\le\mu_{f1}\le 1.05\mu_{f1}^*$, $0.1=\mu_{f3}\le 1$, and $g_j\le 0$, $j=1{,}116$.
[c]Max. $F=\mu_{f3}$ subject to $0.95\mu_{fi}^*\le\mu_{fi}\le 1.05\mu_{fi}^*$, $i=1, 2$, and $g_j\le 0$, $j=1{,}116$.
[d]Lower bound value.

given in Tables 3 and 4. The constraint set in each case is taken as follows:

$$
\begin{aligned}
&0.1 \le \mu_{f_i} \le 1.0, \qquad i = 1, 2, 3\\
&\sigma_{ij} \le \sigma_0, \quad i = 1, 2, \ldots, 25, \quad j = 1, 2\\
&\sigma_{ij} \le \sigma_{ib}, \quad i = 1, 2, \ldots, 25, \quad j = 1, 2\\
&x_i^{(l)} \le x_i \le x_i^{(u)}, \qquad i = 1, 2, \ldots, 8
\end{aligned}
\tag{35}
$$

where σ_{ij} is the stress in member i in load condition j, σ_0 is the maximum permissible stress, σ_{ib} is the buckling stress in member i, $x_i^{(l)}$ is the lower bound on x_i, and $x_i^{(u)}$ is the upper bound on x_i. The data of the problem are taken as $x_i^{(l)} = 0.1$ in.^{2}, $x_i^{(u)} = 5.0$ in.^{2}, $\sigma_0 = 40{,}000$ lb/in.^{2} in both tension and compression, $E = 10^7$ lb/in.^{2}, and $\rho = 0.1$ lb/in.^{3}. The results shown in Table 3 indicate that the simple additive model yields a more balanced design (which corresponds to a weight of 561.1 lb compared with the worst value of 1619.3 lb, a deflection of 1.07 in. compared with the worst value of 1.92 in., and a frequency of 101.6 Hz compared with the worst value of 70.2 Hz). In the weighted sum method, the results indicate a better value to the goal for which a higher weight was assigned. For example, a weight of 271.4 lb compared with the best value of 233.1 lb, a deflection of 0.34 in. compared with the best value of 0.31 in., and a frequency of 107.5 Hz compared with the best value of 108.6 Hz were achieved when the respective goal was given a higher weight in the formulation. The results given by the sum of squares model are similar to those given by the simple additive model except that the structural weight improved slightly at the expense of deflection and natural frequency. In the preemptive priority model, the results (shown in Table 4) were found to be more favorable to weight compared with the other goals. This can be attributed to the objective (weight in this case) considered as the first priority goal.

It can also be observed from Table 4 that the optimum solution at the end of stage (i) corresponds to membership function values of $\mu_{f_1} = 0.993$, $\mu_{f_2} = 0.103$, and $\mu_{f_3} = 0.198$. The value of μ_{f_1} has reduced to 0.948 whereas the values of μ_{f_2} and μ_{f_3} have improved to 0.369 and 0.357, respectively, at the end of stage (iii). Thus μ_{f_2} and μ_{f_3} have improved by 258.2 and 80.3%, respectively, at the expense of 4.5% reduction in μ_{f_1}. The active constraint sets are also different for the solutions found in the various stages.

These results indicate that the simple additive and the sum of squares models can yield results without reflecting a bias to any objective function whereas the weighted sum and the preemptive priority models incorporate a preference structure for the objective functions. Also, the preemptive priority model requires more computational effort compared with the weighted sum models. On the other hand, the weighted sum model may not be able to incorporate the true preference of the designer (because of the difficulty in choosing proper weights) whereas the preemptive priority model is ideally suited for that purpose (since μ_{f_i} are minimized sequentially according to the relative preference of the objective functions).

Conclusion

The fuzzy linear goal programming approach is extended to fuzzy nonlinear goal programming. The relationship between Pareto-optimal solutions and goal programming is also discussed. Several computational models, including the simple additive model, weighted additive model, squared additive model, and the preemptive priority model, are presented for the solution of nonlinear fuzzy goal programming problems. Two numerical examples, one involving the design of a 2-bar truss with 2 goals and the other dealing with the design of a 25-bar truss with 3 goals, are presented to illustrate the computational procedure. Although linear membership functions are used in the numerical examples, the procedure remains the same even for nonlinear membership functions.[17] Methods of handling fuzzy constraints, along with fuzzy goals, are also presented with numerical results. The methodologies presented in this work are expected to aid in the preliminary design of structural and mechanical systems containing imprecise and vague information in the goals and/or constraints.

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